Application of Fuzzy Quantifiers in Image Processing: A Case Study

Ingo Glöckner and Alois Knoll
Technical Computer Science, Faculty of Technology,
University of Bielefeld, 33501 Bielefeld, Germany

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Abstract

Fuzzy quantifiers, i.e. operators intended to provide a numerical interpretation of natural language (NL) quantifiers like 'almost all', are valuable tools for image processing, in particular to express accumulative (second order) properties of fuzzy image regions. However, approaches to fuzzy quantification will unfold their full potential only if the proposed operators capture the meaning of NL quantifiers. We present an exemplary evaluation of one of the most prominent approaches to fuzzy quantification, Yager's OWA approach [7], with respect to its suitability to model NL quantification over fuzzy image regions.

1 Introduction

In developing a system for the content-based retrieval of meteorological documents [5], we have faced the problem of ranking satellite images according to accumulative criteria such as *"almost all of Southern Germany is cloudy"*. In this image ranking task, we have a set E of pixel coordinates. Each pixel $e \in E$ has an associated relevance $\mu_{X_1}(e) \in \mathbf{I} = [0,1]$ with respect to the ranking task, which in this case expresses the degree to which pixel $e \in E$ belongs to Southern Germany, and each pixel has an associated evaluation $\mu_{X_2}(e) \in \mathbf{I}$ which expresses the degree to which the pixel is classified as cloudy (see Fig. 1). The mappings $\mu_{X_1}, \mu_{X_2} : E \longrightarrow \mathbf{I}$ can be viewed as membership functions representing fuzzy subsets $X_1, X_2 \in \widetilde{\mathcal{P}}(E)$ of E, where $\widetilde{\mathcal{P}}(E)$ is the fuzzy powerset of E. Our goal is to determine a mapping $\widetilde{Q} : \widetilde{\mathcal{P}}(E) \times \widetilde{\mathcal{P}}(E) \longrightarrow \mathbf{I}$ which, for each considered satellite image, combines these data to a numerical result $\widetilde{Q}(X_1, X_2) \in \mathbf{I}$ as requested by the NL expression "almost all". The images are then presented in decreasing order of relevance with respect to the search criterion.

Apparently, *an operator which implements "almost all" yields adequate results only if it captures the meaning of "almost all"*. We have therefore decided to base our solution to the image ranking problem on (a) the Theory of Generalized Quantifiers (TGQ [1]), which has developed important concepts for describing the meaning of NL quantifiers; and (b), methods from fuzzy set theory, known as *fuzzy linguistic quantifiers* [8, 6], which are concerned with the use of concepts without sharply defined boundaries ("Southern Germany", "cloudy", "almost all").

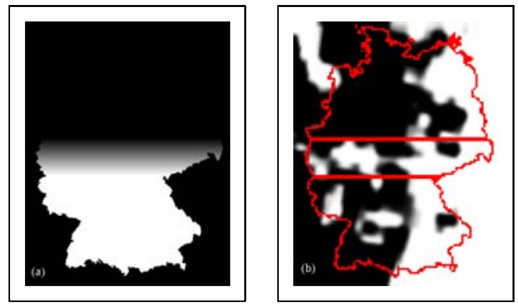


Figure 1: Images for ranking task. (a) A possible definition of X_1 = southern_germany; pixels with $\mu_{X_1}(e) = 1$ depicted white. (b) Fuzzy image region X_2 = cloudy; cloudiness depicted white. The contours of Germany, split in southern, intermediate and northern part, have been added to facilitate interpretation.

2 Two-Valued Generalized Quantifiers

Following TGQ, an n*-ary generalized quantifier* on a base set $E \neq \varnothing$ is a mapping $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{2} = \{0,1\}$ which to each n-tuple of crisp subsets $X_1, \ldots, X_n \in \mathcal{P}(E)$ of E assigns a two-valued quantification result $Q(X_1, \ldots, X_n) \in \mathbf{2}$. Well-known examples are

$$\forall_E(X) = 1 \Leftrightarrow X = E$$
$$\exists_E(X) = 1 \Leftrightarrow X \neq \varnothing$$
$$\mathbf{all}_E(X_1, X_2) = 1 \Leftrightarrow X_1 \subseteq X_2$$
$$\mathbf{some}_E(X_1, X_2) = 1 \Leftrightarrow X_1 \cap X_2 \neq \varnothing$$
$$\mathbf{atleast\ k}_E(X_1, X_2) = 1 \Leftrightarrow |X_1 \cap X_2| \geq k\,.$$

Whenever the base set is clear from the context, we drop the subscript E; $|.|$ is cardinality. For finite E, we can define proportional quantifiers

$$[\textbf{rate} \ge r](X_1, X_2) = 1 \Leftrightarrow |X_1 \cap X_2| \ge r\,|X_1|$$
$$[\textbf{rate} > r](X_1, X_2) = 1 \Leftrightarrow |X_1 \cap X_2| > r\,|X_1|\,.$$

for $r \in \mathbf{I}$, $X_1, X_2 \in \mathcal{P}(E)$. By the *scope* of an NL quantifier we denote the argument occupied by the verbal phrase (e.g. "sleep" in "all men sleep"); by convention, the scope is the last argument of a quantifier. The first argument of a two-place quantifier is called *restriction*, the two-place use of a two-place quantifier *restricted use*, and the one-place use (relative to the whole domain E) *unrestricted use*. E.g., the unrestricted use of **all** : $\mathcal{P}(E)^2 \longrightarrow \mathbf{2}$ is modeled by $\forall : \mathcal{P}(E) \longrightarrow \mathbf{2}$, which has $\forall(X) = \textbf{all}(E, X)$.

3 Fuzzy Generalized Quantifiers

An *n-ary fuzzy quantifier* $\widetilde{Q}$ on a base set $E \neq \varnothing$ is a mapping $\widetilde{Q} : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ which to each n-tuple of fuzzy subsets $X_1, \dots, X_n$ of E assigns a gradual result $\widetilde{Q}(X_1, \dots, X_n) \in \mathbf{I}$.[1] An example is

$$\widetilde{\forall}(X) = \inf_{e \in E} \mu_X(e)\,, \qquad X \in \widetilde{\mathcal{P}}(E)\,.$$

How can we justify which fuzzy quantifier corresponds to a given NL quantifier? Fuzzy quantifiers are possibly too rich a set of operators to investigate this question directly, and all approaches to fuzzy quantification have introduced some kind of *simplified representation*.

4 Fuzzy Linguistic Quantifiers

Following Zadeh [8, 9], most approaches to fuzzy quantification have chosen to define *fuzzy linguistic quantifiers* as fuzzy subsets of the nonnegative reals (absolute type like **some**, with membership functions $\mu_Q \in \mathbf{I}^{\mathbb{R}^+}$), or of the unit interval (proportional type like **most** with $\mu_Q \in \mathbf{I}^{\mathbf{I}}$).[2] These "fuzzy numbers" provide the desired simplified representation. For example, we can define a proportional fuzzy linguistic quantifier **almost all** by $\mu_{\textbf{almost all}}(x) = S(x, 0.7, 0.9)$ for all $x \in \mathbf{I}$, using Zadeh's S-function (see Fig. 2). The μ_Q are not directly applicable to fuzzy sets for quantification purposes. We need a mechanism $\mathcal{Z}$ which maps each μ_Q to fuzzy quantifiers $\mathcal{Z}_E^{(1)}(\mu_Q) : \widetilde{\mathcal{P}}(E) \longrightarrow \mathbf{I}$, to model the unrestricted use relative to E, and $\mathcal{Z}_E^{(2)}(\mu_Q) : \widetilde{\mathcal{P}}(E)^2 \longrightarrow \mathbf{I}$, to model the restricted use, relative to the first argument.[3] Several

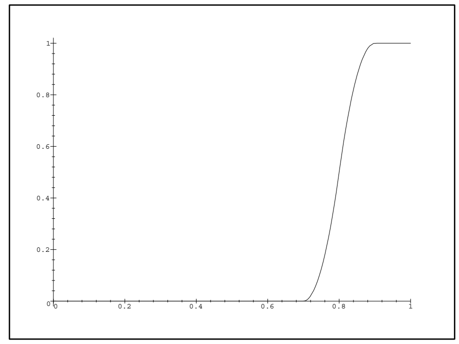


Figure 2: A possible definition of **almost all**

definitions of $\mathcal{Z}$ have been described in the literature (a survey is given in [6]).

5 The OWA approach

Yager [7] models fuzzy quantifiers as Ordered Weighted Averaging (OWA) operators, an approach which has become popular in several applications (see e.g. [2]). Only the proportional type $\mu_Q \in \mathbf{I}^{\mathbf{I}}$ of fuzzy linguistic quantifiers is considered. In addition, μ_Q is assumed to be *regular nondecreasing*, i.e. $\mu_Q(0) = 0$, $\mu_Q(1) = 1$, and $\mu_Q(x) \le \mu_Q(y)$ if $x \le y$. Given a finite base set $E \neq \varnothing$, $m = |E|$ and $\mu_Q \in \mathbf{I}^{\mathbf{I}}$, let us define $\mu_{Q,E} : \{0, \dots, m\} \longrightarrow \mathbf{I}$ by $\mu_{Q,E}(i) = \mu_Q(\frac{i}{m})$, for $i = 0, \dots, m$. Given $X \in \widetilde{\mathcal{P}}(E)$, we denote by $\mu_{[i]}(X) \in \mathbf{I}$, $i = 1, \dots, m$, the i-th largest membership value of X (including duplicates). $\text{OWA}^{(1)}_{\text{prp}}(\mu_Q) : \widetilde{\mathcal{P}}(E) \longrightarrow \mathbf{I}$ is defined by

$$\text{OWA}^{(1)}_{\text{prp}}(\mu_Q)(X) = \sum_{i=1}^{m} (\mu_{Q,E}(i) - \mu_{Q,E}(i-1)) \cdot \mu_{[i]}(X)\,,$$

(where μ_Q is regular nondecreasing). In order to model two-place quantifiers, Yager introduces a weighting formula parametrised by the so-called *degree of orness*,

$$\text{orness}(\mu_{Q,E}) = \frac{1}{m-1} \sum_{i=1}^{m} (m-i)(\mu_{Q,E}(i) - \mu_{Q,E}(i-1))\,.$$

In other words, $\text{orness}(\mu_{Q,E}) = \frac{1}{m-1} \sum_{i=1}^{m-1} \mu_{Q,E}(i)$. The weighting function $g_\alpha : \mathbf{I} \times \mathbf{I} \longrightarrow \mathbf{I}$ corresponding to a regular nondecreasing μ_Q with $\alpha = \text{orness}(\mu_{Q,E})$, is defined by $g_\alpha(x_1, x_2) = (x_1 \vee (1-\alpha)) \cdot {x_2}^{x_1 \vee \alpha}$, for all $x_1, x_2 \in \mathbf{I}$.[4] For $X_1, X_2 \in \widetilde{\mathcal{P}}(E)$, we define $g_\alpha(X_1, X_2) \in \widetilde{\mathcal{P}}(E)$ pointwise by $\mu_{g_\alpha(X_1, X_2)}(e) =$

[1] This definition closely resembles Zadeh's [9, pp.756] alternative view of fuzzy quantifiers as fuzzy second-order predicates.

[2] A^B denotes the set of mappings $f : B \longrightarrow A$.

[3] We will use the superscripts only when necessary to discern the unrestricted and restricted use, and usually drop the subscript E.

[4] We shall assume that $0^0 = 1$, i.e. $g_0(0, 0) = 1$, in order to have $\text{OWA}^{(2)}_{\text{prp}}(\mu_\forall)(\varnothing, \varnothing) = 1$, where $\mu_\forall(1) = 1$, $\mu_\forall(x) = 0$ else.

$g_\alpha(\mu_{X_1}(e), \mu_{X_2}(e))$, for all $e \in E$.
$\mathrm{OWA}^{(2)}_{\mathrm{prp}}(\mu_Q) : \widetilde{\mathcal{P}}(E)^2 \longrightarrow \mathbf{I}$ is then defined by

$$\mathrm{OWA}^{(2)}_{\mathrm{prp}}(\mu_Q) = \mathrm{OWA}^{(1)}_{\mathrm{prp}}(\mu_Q)(g_\alpha(X_1, X_2)),$$

for all $X_1, X_2 \in \widetilde{\mathcal{P}}(E)$. Let us note that the degree of orness, and hence $\mathrm{OWA}^{(2)}_{\mathrm{prp}}$, is *undefined* if $|E| = 1$.

6 An Evaluation Framework Based on TGQ

How can we evaluate the linguistic adequacy of the OWA approach? We first need a simplified representation of fuzzy quantifying operators in which all two-valued quantifiers of TGQ can be embedded, and which allows for a straightforward generalisation of the concepts developed by TGQ to describe the meaning of quantifiers.
An *n-ary semi-fuzzy quantifier* on a base set $E \neq \varnothing$ is a mapping $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ which to each n-tuple of *crisp* subsets of E assigns a gradual result $Q(X_1, \dots, X_n) \in \mathbf{I}$. Semi-fuzzy quantifiers are half-way between two-valued quantifiers and fuzzy quantifiers because they have crisp input, fuzzy (gradual) output. Every two-valued quantifier of TGQ is a semi-fuzzy quantifier by definition. We shall call $Q : \mathcal{P}(E) \times \mathcal{P}(E) \longrightarrow \mathbf{I}$ *proportional* if defined by

$$Q(X_1, X_2) = \begin{cases} f\left(\frac{|X_1 \cap X_2|}{|X_1|}\right) & : \quad X_1 \neq \varnothing \\ c & : \quad \text{else} \end{cases} \qquad (1)$$

where $f : \mathbf{I} \longrightarrow \mathbf{I}$ and $c \in \mathbf{I}$. For **almost all** we can have $f = \mu_{\textbf{almost all}}$ and $c = 1$. It is relatively easy to understand the input-output behavior of a semi-fuzzy quantifier, which is stated in terms of crisp argument sets. However, semi-fuzzy quantifiers do not accept fuzzy input, and we have to make use of a fuzzification mechanism which transports these to fuzzy quantifiers.

A *quantifier fuzzification mechanism* (QFM) $\mathcal{F}$ assigns to each semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ a corresponding fuzzy quantifier $\mathcal{F}(Q) : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ of the same arity n and on the same base set E. Existing approaches like OWA cannot be directly viewed as QFMs because they are not applicable to semi-fuzzy quantifiers. However, given such a mechanism $\mathcal{Z}$, we can often reconstruct a partially defined QFM $\mathcal{F}$ as follows. The *underlying semi-fuzzy quantifier* $\mathcal{U}(\widetilde{Q}) : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ of a given fuzzy quantifier $\widetilde{Q} : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ is defined $\mathcal{U}(\widetilde{Q}) = \widetilde{Q}|_{\mathcal{P}(E)^n}$, i.e. $\mathcal{U}(\widetilde{Q})(X_1, \dots, X_n) = \widetilde{Q}(X_1, \dots, X_n)$ for all *crisp* arguments $X_1, \dots, X_n \in \mathcal{P}(E)$. Provided the membership function μ_Q of a fuzzy linguistic quantifier, we obtain the corresponding semi-fuzzy quantifier (relative to $\mathcal{Z}$) as $Q = \mathcal{U}(\mathcal{Z}(\mu_Q))$, and use this to define $\mathcal{F}(Q) = \mathcal{Z}(\mu_Q)$. The construction of $\mathcal{F}$ succeeds only if $\mathcal{U}(\mathcal{Z}(\mu_Q)) \mapsto \mathcal{Z}(\mu_Q)$ is functional. We shall call this the *quantifier framework assumption* (QFA). We say that $\mathcal{Z}$ can *represent* a semi-fuzzy quantifier Q if there exists μ_Q such that $Q = \mathcal{U}(\mathcal{Z}(\mu_Q))$, i.e. $\mathcal{F}$ is defined on Q.

By viewing approaches to fuzzy quantification as instances of (partially defined) QFMs, we can now judge the adequacy of these approaches by investigating preservation properties of the corresponding partial QFMs. A comprehensive account of such adequacy conditions is given in [3, 4]. The most basic requirement on $\mathcal{F}$ is *correct generalisation*: for all $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ on which $\mathcal{F}$ is defined, $\mathcal{U}(\mathcal{F}(Q)) = Q$, i.e. $\mathcal{F}(Q)$ coincides with Q on crisp arguments. This is ensured by our construction of $\mathcal{F}$ from $\mathcal{Z}$.
The *negation* $\neg Q$ and *antonym* $Q\neg$ of Q are defined by

$$\neg Q(X_1, \dots, X_n) = 1 - Q(X_1, \dots, X_n)$$
$$Q\neg(X_1, \dots, X_n) = Q(X_1, \dots, X_{n-1}, \neg X_n)$$

($\neg X$ is complementation); the *dual* of Q is $\neg Q \neg$. We say that $\mathcal{F}$ preserves negation, antonymy, and dualisation, if $\mathcal{F}(\neg Q) = \neg \mathcal{F}(Q)$, $\mathcal{F}(Q\neg) = \mathcal{F}(Q)\neg$ and $\mathcal{F}(\neg Q \neg) = \neg \mathcal{F}(Q) \neg$, respectively.
One of the pervasive properties of NL quantifiers is conservativity. We shall call $Q : \mathcal{P}(E)^2 \longrightarrow \mathbf{I}$ *conservative* if $Q(X_1, X_2) = Q(X_1, X_1 \cap X_2)$ for all $X_1, X_2 \in \mathcal{P}(E)$. Conservativity expresses that an element of the domain which is irrelevant to the restriction of the quantifier does not influence the quantification result at all. A corresponding fuzzy quantifier $\widetilde{Q} : \widetilde{\mathcal{P}}(E)^2 \longrightarrow \mathbf{I}$ should at least have $\widetilde{Q}(X_1, X_2) = \widetilde{Q}(X_1, \mathrm{spp}(X_1) \cap X_2)$, where $\mathrm{spp}(X) = \{x \in E : \mu_X(e) > 0\}$. In our image ranking task, then, a pixel $e \in E$ which does not belong to Southern Germany, i.e. $\mu_{X_1}(e) = 0$, will not affect the result of “almost all X_1 are X_2”, as is highly desirable.

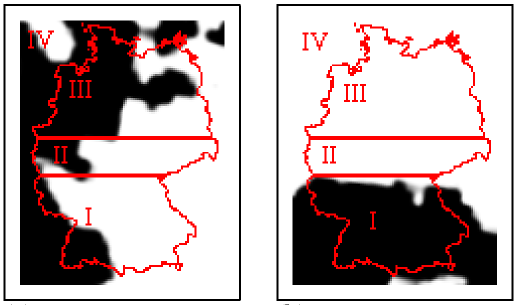
7 Evaluation of the OWA Approach

It is easily observed that the OWA approach fulfills the QFA. But there is negative evidence as regards its adequacy. Firstly, if $\alpha = \mathrm{orness}(\mu_{Q,E}) \in (0, 1)$,[5] then

$$\mathrm{OWA}^{(2)}_{\mathrm{prp}}(\mu_Q)(\varnothing, \varnothing) = 0 \neq (1 - \alpha) = \mathrm{OWA}^{(2)}_{\mathrm{prp}}(\mu_Q)(\varnothing, E),$$

which shows that, except **all** and **some**, no semi-fuzzy quantifier $Q : \mathcal{P}(E)^2 \longrightarrow \mathbf{I}$ which can be represented by OWA is conservative. In particular, *the OWA approach cannot represent any proportional semi-fuzzy quantifiers except for* **all** *and* **some**.[6] But this is exactly the type of quantifiers OWA is intended to model.

[5] i.e. μ_Q is distinct from the universal and the existential quantifier
[6] all proportional quantifiers are conservative by Eq. (1).



(a) desired: 1, OWA: 0.1 (b) desired: 0, OWA: 0.6

Figure 3: At least 60 percent of Southern Germany are cloudy

If we still try to use OWA for interpreting proportional quantifiers, implausible results must be expected. For example, let us consider $\mu_{[\text{rate}\geq 0.6]}$, defined by

$$\mu_{[\text{rate}\geq 0.6]}(x) = \begin{cases} 1 & : \quad x \geq 0.6 \\ 0 & : \quad \text{else} \end{cases}$$

for evaluating "at least 60 percent of Southern Germany are cloudy" (see Fig. 3). In situation (a), we expect the result 1, because sufficiently many pixels which fully belong to Southern Germany (I) are classified as fully cloudy that, regardless of whether we view the intermediate cases (II) as belonging to Southern Germany or not, its cloud coverage is always larger than 60 percent. Likewise in (b), we expect a result of 0 because regardless of whether the pixels in (II) are viewed as belonging to Southern Germany, its cloud coverage is always smaller than 60 percent. OWA, however, ranks image (b) much higher than image (a). This counterintuitive result is explained by OWA's lack of conservativity: the cloudiness degrees of pixels in areas (III) and (IV), which do not belong to Southern Germany at all, still have a strong (and undesirable) impact on the computed results.

Further problems arise from the fact that $\text{OWA}^{(2)}_{\text{prp}}$ does not preserve duals. Suppose we wish to evaluate the criterion "less than 60 percent of Southern Germany are cloudy". Because this quantifier is not regular nondecreasing, we must resort to one of the following equivalent statements:

i. "It is not the case that atleast 60 percent of Southern Germany are cloudy", i.e. use negation and compute $\neg\text{OWA}^{(2)}_{\text{prp}}(\mu_{[\text{rate}\geq 0.6]})(X_1, X_2)$;

ii. "More than 40 percent of Southern Germany are not cloudy", i.e. use the antonym and compute $\text{OWA}^{(2)}_{\text{prp}}(\mu_{[\text{rate}>0.4]})(X_1, \neg X_2)$.

Unfortunately, these statements are not equivalent when using OWA; when applied to the images in Fig. 3, we obtain different results as shown in Table 1. In this case, both i. and ii. compute the same (wrong) ranking, which shows that neither alternative is correct; in other cases, their rankings can differ. The problem is that OWA, which cannot model "less than 60 percent" directly, forces us to choose one of i., ii.; but due to their expected equivalence, there is no preference for either choice. In an appropriate model, the results of i. and ii. coincide, making this choice inessential; OWA, however, fails to preserve the required duality of "atleast 60 percent" vs. "more than 40 percent".

Quantifier	**Fig. 3(a)**	**Fig. 3(b)**
$\neg\text{OWA}^{(2)}_{\text{prp}}(\mu_{[\text{rate}\geq 0.6]})$	0.9	0.4
$\text{OWA}^{(2)}_{\text{prp}}(\mu_{[\text{rate}>0.4]})\neg$	0.4	0
(desired result)	0	1

Table 1: Less than 60 percent of Southern Germany are cloudy

8 Conclusion

Fuzzy quantifiers can prove useful in image processing if these operators are linguistically adequate. However, our findings indicate that OWA does not pass the adequacy test. Our results on other approaches based on fuzzy linguistic quantifiers are also negative [4]. In order to adequately perform the image ranking task, we have decided to abandon fuzzy linguistic quantifiers, in favor of a theory directly built on semi-fuzzy quantifiers and QFMs. DFS theory [3] specifies an axiom set which includes (or entails) the adequacy conditions presented here. A QFM which satisfies these axioms avoids counterintuitive results like those of OWA from the outset.

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